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A FAST FOURIER TRANSFORM TECHNIQUE AND ITS APPLICATION TO FOURIER SPECTROSCOPY (1)

M. L. FORMAN

Concord Radiance Laboratory of Utah State University Davis Road
Bedford, Massachusetts, U. S. A.

Résumé. — Description d'une méthode nouvelle, issue des travaux de Good et de Cooley et Tukey pour le calcul des transformées de Fourier. Le temps de calcul est proportionnel à $N \operatorname{Log}_2 N$ si N est le nombre de points d'entrée et de sortie. Des essais avec un ordinateur IBM 7044 ont permis de faire la transformée de $N = 4\,096$ points en 14 secondes.

Abstract. — A description of a new method for computing Fourier transforms is given ; it was originated by Good and by Cooley and Tukey. The computation time is proportional to $N \operatorname{Log}_2 N$ where N is the number of inputs. With an IBM 7044 computer a computation time equal to 14 seconds has been obtained for $N = 4\,096$.

The problem of computing Fourier transforms on digital computers is of great concern to many people, especially those involved in Fourier Spectroscopy [1-3]. An efficient method developed by J. W. Cooley and J. W. Tukey [4] for the computation of complex Fourier transforms on digital computers and its application to Fourier spectroscopy is discussed. The time of computation is proportional to $N \log_2 N$, N being the number of input and output points.

In 1958, I. J. Good [5] developed a technique for computing Fourier transforms based on an interaction algorithm for t^n factorial experiments ; however, this method remained obscured to the view of optical physicists, radio astronomers, et al, due to the choice of journal. Recently, Cooley and Tukey have presented an interpretation of Good's methods ; however, the frequency components come out in a jumbled (binary bit inverted) order. More recently, at a seminar given by T. G. Stockham, Jr., and C. M. Rader of M. I. T., this method was brought to my attention with a technique for obtaining frequency components in the correct order.

This letter is meant to bring these computational techniques to the attention of those working in Fourier optics.

Fast Transform Method. — It is desired to compute the Fourier transform of N equally spaced points at N frequencies from 0 to $2\pi(1 - 1/N)$ radians/sec in steps of $2\pi/N$. Neglecting constants we can write

$$F(k) = \sum_{j=0}^{N-1} S(j)\, W^{jk} \tag{1}$$

where

$$W = \exp(i\, 2\pi/N)\,.$$

Let us now consider a binary number representation of j and k, that is,

$$j = (j_{n-1}, j_{n-2}, \ldots, j_0) = \sum_{i=0}^{n-1} j_i\, 2^i \tag{2a}$$

$$k = (k_{n-1}, k_{n-2}, \ldots, k_0) = \sum_{i=0}^{n-1} k_i\, 2^i \tag{2b}$$

where j_i and k_i take on the values 0 and 1 and consider the product jk. Hence

$$\begin{array}{r}
(k_{n-1}, k_{n-2}, \ldots, k_1, k_0) \\
\underline{(j_{n-1}, j_{n-2}, \ldots, j_1, j_0)} \\
(k_{n-1}, k_{n-2}, \ldots, k_1, k_0)\, j_0 \\
k_{n-1}(k_{n-2}, \ldots\ldots\ldots, k_1, k_0, 0)\, j_1 \\
k_{n-1}, \ldots, k_1(k_0, \ldots\ldots\ldots\ldots\ldots, 0, 0)\, j_{n-1}\,.
\end{array}$$

For any term shown above, if k_i is outside of the parenthesis, the numerical value associated with that term will be divisible by 2^n. Thus, if we restrict our-

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selves to $N = 2^n$, W^{jk} may be factored into n factors, that is,

$$W^{jk} = W^{(k_{n-1},k_{n-2},\dots,k_0)j_0}\, W^{(k_{n-2},\dots,k_0,0)j_1} \dots$$
$$\dots W^{(k_0,0\dots,0)j_{n-1}} . \qquad (3)$$

Provided the above restriction on N is met, eqn (1) can be written as

$$F(k_{n-1}, \dots, k_0) = \sum_{j_{n-1}=0}^{1} \sum_{j_{n-2}=0}^{1} \dots$$
$$\dots \sum_{j_0=0}^{1} S(j_{n-1}, j_{n-2}, \dots, j_0) \dots W^{(k_0,0,\dots,0)j_{n-1}} \dots$$
$$\dots W^{(k_{n-1},k_{n-2},\dots,k_0)j_0} . \qquad (4)$$

Considering the sum over j_{n-1}, we may define

$$S_0(k_0, j_{n-2}, \dots, j_0) \equiv \sum_{j_{n-1}=0}^{1} S(j_{n-1}, \dots, j_0) \times$$
$$\times\ W^{(k_0,0,\dots,0)j_{n-1}} . \qquad (5a)$$

Substituting the above in eqn (4) and summing on j_{n-2}, we can write

$$S_1(k_1, k_0, j_{n-3}, \dots, j_0) \equiv$$
$$\equiv \sum_{j_{n-2}=0}^{1} S_0(k_0, j_{n-2}, \dots, j_0)\, W^{(k_1,k_0,0,\dots,0)j_{n-2}} . \qquad (5b)$$

The following recursion relationship may then be obtained

$$S_m(k_m, \dots, k_0, j_{n-m-2}, \dots, j_0) \equiv$$
$$\equiv \sum_{j_{n-m-1}=0}^{1} S_{m-1}(k_{m-1}, \dots, k_0, j_{n-m-1}, \dots, j_0) \times$$
$$\times\ W^{(k_m,\dots,k_0,0,\dots,0)j_{n-m-1}} . \qquad (5c)$$

As there are only n terms within the parenthesis the n-th summation yields the Fourier transform, that is

$$F(k_{n-1}, \dots, k_0) = S_{n-1}(k_{n-1}, \dots, k_0)\ ({}^{1}) . \qquad (5d)$$

Since N frequencies are desired, one need only make Nn complex multiplications and additions. A direct application of equations (5*a*, 5*b* and 5*c*) will be made for $N = 8$; $j = j_2, j_1, j_0$; $k = k_2, k_1, k_0$.

$$S_0(k_0, j_1, j_0) = S(0, j_1, j_0) + S(1, j_1, j_0)\, W^{(k_0,0,0)}$$
$$S_1(k_1, k_0, j_0) = S(0, 0, j_0) + S(1, 0, j_0)\, W^{(k_0,0,0)}$$
$$+\ S(0, 1, j_0)\, W^{(k_1,k_0,0)}$$
$$+\ S(1, 1, j_0)\, W^{(k_0,0,0)}\, W^{(k_1,k_0,0)}$$
$$S_2(k_2, k_1, k_0) = S(0, 0, 0) + S(1, 0, 0)\, W^{(k_0,0,0)}$$
$$+\ S(0, 1, 0)\, W^{(k_1,k_0,0)}$$
$$+\ S(1, 1, 0)\, W^{(k_0,0,0)}\, W^{(k_1,k_0,0)}$$
$$+\ S(0, 0, 1)\, W^{(k_2,k_1,k_0)}$$
$$+\ S(1, 0, 1)\, W^{(k_0,0,0)}\, W^{(k_2,k_1,k_0)}$$
$$+\ S(0, 1, 1)\, W^{(k_1,k_0,0)}\, W^{(k_2,k_1,k_0)}$$
$$+\ S(1, 1, 1)\, W^{(k_0,0,0)}\, W^{(k_1,k_0,0)}\, W^{(k_2,k_1,k_0)}$$

Thus, if we consider the term whose fundamental is $3\,\pi/4$ or $k = 0, 1, 1$, we find

$$F(0, 1, 1) = S_0 + S_1\, W^3 + S_2\, W^6 + S_3\, W^1$$
$$+\ S_4\, W^4 + S_5\, W^7 + S_6\, W^2 + S_7\, W^5$$

where the Arabic equivalent of the binary numbers are used, modulo 8, and the terms have been reordered.

Application to Fourier Spectroscopy. — It is clear that the Cooley-Tukey method of computing Fourier transforms is applicable to Fourier spectroscopy. However, certain modifications to the input data are necessary.

If one has M data points where $M \neq 2^n$, it is possible to augment the data with L zeros such that

$$M + L = 2^n = N .$$

This technique can also be used in order to obtain more output points per spectral resolution element if one starts with less points than the maximum number allowed by the program. If Δx cm is the sampling interval of the points of the interferogram, the frequency interval between adjacent points in the spectrum will be

$$\Delta\sigma = 1/N\,\Delta x\ \text{cm}^{-1} \qquad (6)$$

which in the symmetric case is equivalent to the resolution of the unapodized record. When Δx is

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(1) It is of interest to note that in practice the method described by Cooley and Tukey (the binary bit inverted) yields times about 30 % faster than those given here. The binary bit inverted algorithm may be derived from the left hand side of equations 5*b* and 5*c*. $S_1(k_1, k_0, j_{n-3}, \dots, j_0)$ of equation 5*b* is an arbitrary definition and may just as well be written

$$S_1(k_0, k_1, j_{n-3}, \dots, j_0) .$$

Hence, the general recursion formule, equation 5*c* will become

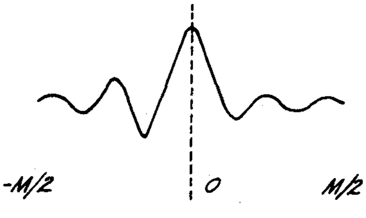
$$S_m(k_0, \dots, k_m, j_{n-m-2}, \dots, j_0) \equiv$$
$$\equiv \sum_{j_{n-m-1}=0}^{1} S_{m-1}(k_0, \dots, k_{m-1}, j_{n-m-1}, \dots, j_0) \times$$
$$\times\ W^{(k_m,\dots,k_0,0,\dots,0)j_{n-m-1}} .$$

After the n-th summation, equation 5*d* becomes

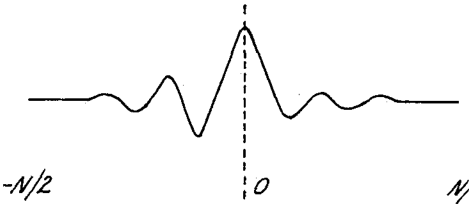
$$F(k_{n-1}, \dots, k_1, k_0) \equiv S_{n-1}(k_0, k_1, \dots, k_{n-1}) .$$

near the sampling theory limit, all the output points will contain useful spectral information. However, in this case one obtains one output point per resolution element, and spectral interpolation techniques discussed by Filler [6] should be applied to obtain a better picture of the spectrum. If one oversamples the data, the efficiency of the technique is decreased as several frequency components will be outside of the optical region of interest.

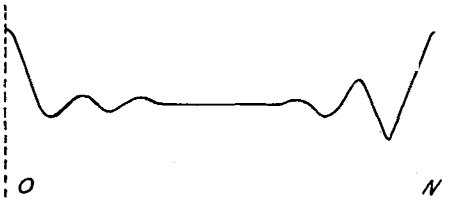
The final transform will have a real and imaginary part. If one has a symmetric interferogram, or forces an asymmetric interferogram to be symmetric by the technique developed by Forman, Steel and Vanasse [7], the real part of the transform is the desired spectrum, the imaginary part being the Hilbert transform. In this case, the zero retardation point should be divided by two. For the asymmetric case, one need only compute the magnitude spectrum ; however, the phase of the frequency components at the center of the interferogram cannot be computed if the interferogram is treated as a unit as the shift theorem will be invoked. The phase information is recoverable if the interferogram is split into two parts and placed in core storage as depicted in figure 1 (^{2}). This is due to the fact that for a specific harmonic σ_0, the first point is multiplied by the sine or cosine of zero degree, the second and N-th points are multiplied by the sine and cosine of plus and minus $2\pi\sigma_0 \Delta x$ respectively, etc.



ORIGINAL INTERFEROGRAM



AUGMENTED INTERFEROGRAM – NO PHASE RETRIEVAL



SET UP FOR PHASE RETRIEVAL

FIG. 1. — This shows how an asymmetric interferogram may be split up in order to retrieve phase information including augmenting by zeros.

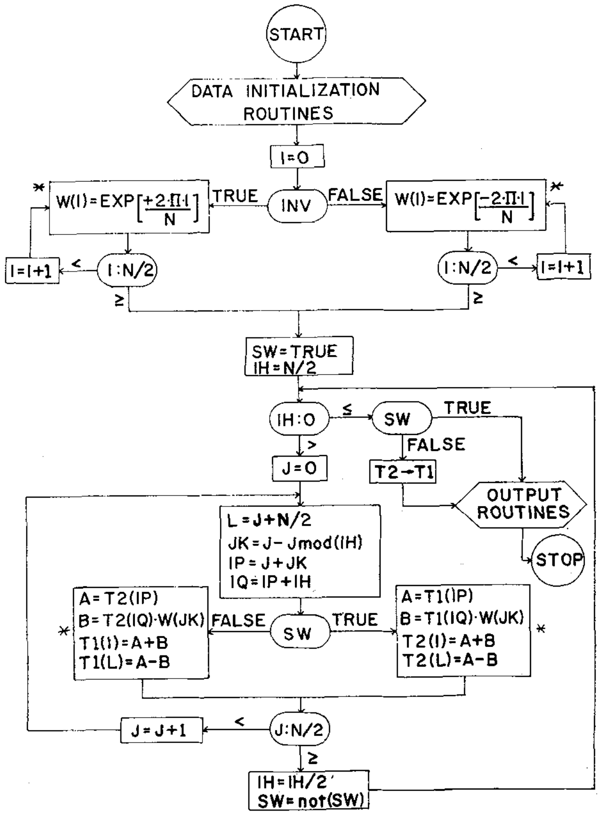


FIG. 2. — Flow chart for 4 906 input points as an aid to programming for obtaining Fourier coefficients.

* indicates complex arithmetic,
: indicates comparison,
→ indicates transfer of data.

Figures 2 and 3 are flow charts for digital computer programs capable of transforming 4 096 or 8 192 input points respectively, the latter being about 50 percent slower. This restriction on points is due to the fact that complex arithmetic requires twice as much storage as normal arithmetic and most computers have 32 777 storage registers of which 28 000 may be available for data storage. This poses a slight limitation on the method ; however, by using mathe-

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(^{2}) A linear shift technique may also be used to recover the phase.

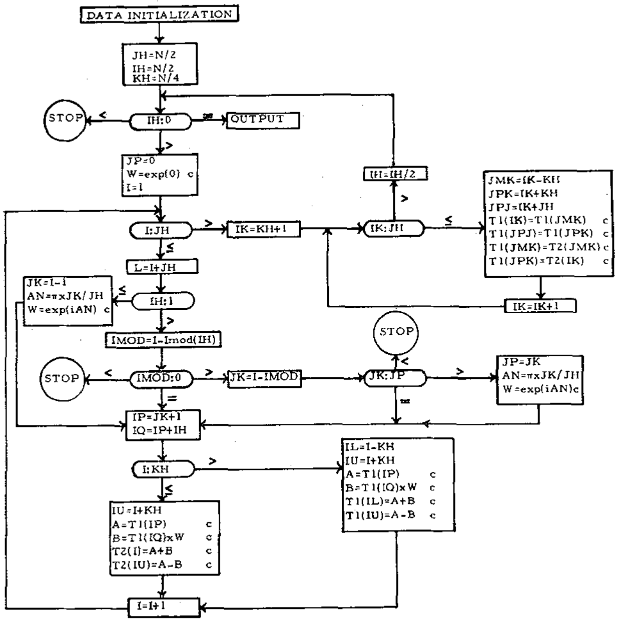

FIG. 3. — Flow chart for 8 192 input points as an aid to programming for obtaining Fourier coefficients.
: indicates comparison,
c indicates complex arithmetic.

matical filtering techniques suggested by Connes and Nozal [8], and suitably selecting the bandwidth of the filters, interferograms with more points than shown above can be reduced in spectral sections.

Results. — The fast transform was programmed for an IBM 7044 at AFCRL, Lexington, Massachusetts and the time of actual transformation was compared to a conventional transform program using Chebeshov Polynomial approximations for the sine and cosine terms. Provided one samples efficiently, the gains in computation time of the actual Fourier transform by the fast transform method increase as N increases. Table 1 gives the time of N input and N output points by both methods. Round-off error is less in the Cooley-Tukey method, as there are far less multiplications and additions.

TABLE 1. — Comparison of times of transformation.

$N =$	512	1 024	2 048	4 096
Conventional	43	171	684	2 734 seconds
Fast	1.35	3	6.6	14.4 seconds

Acknowledgement. — I would like to take this opportunity to express my gratitude to Dr. Thomas G. Stockham, Jr. of the Massachusetts Institute of Technology for bringing the Cooley-Tukey method to my attention, and several discussions on the fine points of the method.

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