Kamenev-Type Oscillation Criteria for Hyperbolic Nonlinear Neutral Delay Difference Equations

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Abstract: By means of the Riccati transformation techniques, we will establish some new oscillation criteria for hyperbolic nonlinear neutral delay difference equations. Our results can be considered as the discrete analogues of the Kamenev-type and Philos-type oscillation criteria.

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1 Introduction

Qualitative theory for discrete dynamic systems with one dimension, that is, for ordinary difference equations which parallels the qualitative theory of differential equations, has been investigated by several authors, see e.g., the monographs [1, 2, 3, 9] and the references cited therein. On the other hand, though the nonlinear discrete dynamics systems involve functions of two or more independent variables, that is partial difference equations (PDEs), are as important as difference equations, comparatively few papers have been devoted to the oscillation theory of their solutions, see, e.g., the papers by Saker [21], Saker and Wong [22] and Zhang [26] and the references cited therein. In fact, partial difference equations arise in the approximation of solutions of partial differential equations by finite difference methods, random walk problems, the study on molecular orbits , mathematical physics problems and other problems in population dynamics [5, 6, 9, 11, 23, 27]. Hence, to further develop the qualitative theory of partial difference equations, in this paper we shall consider the following hyperbolic nonlinear neutral delay difference equation

$$\Delta_2(a_n\Delta_2(y_{m,n}+p_ny_{m,n-\tau})+q_{m,n}f(y_{m,n-\sigma})=r_nLy_{m,n}+\sum_{j\in J}R_{j,n}Ly_{m,n-\gamma_j}, \quad (1)$$

$\{y_{m,n}\}=\{y_{m_1,m_2,...,m_l,n}\}$ which is defined in $\Omega\times\mathbf{N}_{n_0}$, $\mathbf{J}=\{1,2,...,J_0\}$, $\mathbf{N}_{n_0}=\{n_0,n_0+1,...\}$,

$$\Omega=\mathbf{N}(1,N_1)\times\mathbf{N}(1,N_2)\times...\mathbf{N}(1,N_l), \quad (2)$$

[1]The results in this paper have been finshed when the author was at the Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznan, Poland.

where $\mathbf{N}(a,b)=\{a,a+1,...,b\}$ and $Ly_{m,n}$ is the discrete Laplacian operator defined by

$$Ly_{m,n}=\sum_{i=1}^{l}\Delta^2_{m_i}y_{m_1,m_2,...,m_{i-1},m_i-1,m_{i+1}...,m_l,n} \tag{3}$$

where Δ_i^2 is the partial difference operator of order two i.e., $\Delta_i^2 y_{m,n}=\Delta_i(\Delta_i y_{m,n})$, $\Delta_1 y_{m,n}=y_{m+1,n}-y_{m,n}$ and $\Delta_2 y_{m,n}=y_{m,n+1}-y_{m,n}$.

We assume throughout this paper that:

(h1) $a_n, r_n, p_n \in \mathbf{N}_{n_0}\to\mathbf{R}^+$, $\sum\limits_{n=n_0}^{\infty}\frac{1}{a_n}=\infty$, $0\leq p_n<1$, $R_{j,n}\in\mathbf{J}\times\mathbf{N}_{n_0}\to\mathbf{R}^+$;

(h2) $q_{m,n}\in\Omega\times\mathbf{N}_{n_0}\to\mathbf{R}^+$, and there exists a sequence $\{q_n\}$ such that $q_n=\min_{m\in\Omega}\{q_{m,n}\}$, $n\in\mathbb{N}_{n_0}$, and $\{q_n\}$ has a positive subsequence;

(h3) $\sigma,\ \tau\in\mathbf{N}_1$, $\gamma_j\in\mathbf{J}\to\mathbf{N}_1$;

(h4) $f\in C(R,R)$ is convex, $uf(u)>0$ for $u\neq 0$ and $f(u)/u\geq k>0$.

Consider the initial boundary value problem (IBVP) (1.1) with the boundary conditions

$$y_{m,n}=0,\quad m\in\partial\Omega,\ n\in\mathbf{N}_{n_0}, \tag{4}$$

where

$$\begin{aligned}\partial\Omega &= \bigcup_{i=1}^{l}\{(m_1,...,m_{i-1},0,m_{i+1},...,m_l),(m_1,...,m_{i-1},N_i+1,m_{i+1},...,m_l)\},\\ m_i &\in \mathbf{N}(1,N_i),\ 1\leq i\leq l.\end{aligned} \tag{5}$$

and

$$\begin{aligned}\Delta_{m_i}y_{m,n}|_{m_i=0}-g_{m,n}\ y_{m,n}|_{m_i=0} &= 0,\\ \Delta_{m_i}y_{m,n}|_{m_i=N_i}+g_{m,n}\ y_{m,n}|_{m_i=N_i+1} &= 0,\ m\in\Omega,\ n\in\mathbf{N}_{n_0},\ 1\leq i\leq l.\end{aligned} \tag{6}$$

and the initial condition (IC)

$$y_{m,s}=\mu_{m,s},\qquad \text{for } n_0-M\leq s\leq n_0, \tag{7}$$

where $g_{m,n}\geq 0$ for m$\in\Omega$, $n\in\mathbf{N}_{n_0}$, and $M=max\{\sigma,\ \tau,\ \gamma_j\}$.

By a solution of initial boundary value problem (1.1), (B1), [or (B2)] and (1.5) (for short IBVPB1) [for short IBVPB2] we mean a sequence $\{y_{m,n}\}$ which satisfies Eq.(1.1) for $(m,n)\in\Omega\times\mathbf{N}_{n_0}$, satisfies (B1) [B2] for $(m,n)\in\partial\Omega\times\mathbf{N}_{n_0}$ and satisfies IC (1.5) for $(m,n)\in\Omega\times\{n_0-M,...,n_0\}$.

In the continuous case, the differential equation

$$x^{''}(t) + q(t)f(x(t)) = 0, \ \text{t} \geq t_0 \tag{8}$$

has been tackled by many authors, see the survey papers [8, 24, 25] which give over 350 references. It is known that, due to Kamenev [7], the average function $A_\lambda(t)$ defined by

$$A_\lambda(t) = \frac{1}{t^\lambda}\int_{t_0}^{t}(t-s)^\lambda q(s)ds, \quad \lambda \geq 1, \tag{9}$$

plays a crucial role in the oscillation of Eq.(1.6). Philos [19] further improves Kamenev's result by proving the following: Suppose there exist continuous functions H, $h : D \equiv \{(t,s) : t \geq s \geq t_0\} \to \mathbf{R}$ such that (i) $H(t,t) = 0$, $t \geq t_0, (ii)$ $H(t,s) > 0$, $t > s \geq t_0$, and H has a continuous and nonpositive partial derivative on D with respect to the second variable and satisfies

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \geq 0.$$

Further, suppose that

$$\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right]ds = \infty. \tag{10}$$

Then every solution of equation (1.6) oscillates. However, these criteria can not be applied in the case when q(t)=α/t^2 (see [24]). For oscillation of different types of partial delay and neutral delay differential equations we refer to the results in [12, 14-18, 20].

Our aim in this paper, by means of Riccati transformation techniques, we will establish some new oscillation criteria for IBVPB1 and IBVPB2 in $\Omega \times \mathbf{N}_{n_0}$, in the sense that there does not exist $n_1 \in \mathbf{N}_{n_0}$ such that $y_{m,n} > 0$ or $y_{m,n} < 0$ for $n \in \mathbf{N}_{n_1}$. Our Kamenev-type criteria not only the discrete analogues of (1.7) and (1.8) but also improve them and can be applied in the case $q_{m,n} \geq \alpha/n^2$.

2 Oscillation of IBVPB1

In this section we will establish some new oscillation criteria for IBVPB1. Before stating our main results we need the following lemmas.

Lemma 2.1 [4]. *Consider the eigenvalue problem*

$$\left.\begin{array}{ll} Ly_m + \alpha y_m = 0 & m \in \Omega, \\ y_m = 0 & m \in \partial\Omega. \end{array}\right\},$$

where Ω, L and $\partial\Omega$ are defined in (1.2), (1.3) and (1.4) respectively. Let α_0 be the least eigenvalue of problem (2.1) and ϕ_m be the corresponding eigenfunction. Then $\alpha_0 > 0$ and $\phi_m > 0$ $m\in \Omega$.

Lemma 2.2 [3]. (Discrete Green's formula). *Let y_m, z_m be two sequences defined on $\overline{\Omega} \equiv \mathbf{N}(0, N_1) \times \mathbf{N}(1, N_2) \times ...\mathbf{N}(1, N_l + 1)$. Then*

$$\sum_{m\in\Omega} z_m L y_m - \sum_{m\in\Omega} y_m L z_m = \sum_{i=1}^{l} \left\{ \sum_{m_1=1}^{N_1} ... \sum_{m_i=1}^{N_{i-1}} \sum_{m_{i+1}=1}^{N_{i+1}} ... \sum_{m_l=1}^{N_l} [z_m \Delta_{m_i} y_m - y_m \Delta_{m_i} z_m]_{m_i=0}^{N_i} \right\}$$

where Ω and L are defined in (1.2) and (1.3) respectively.

Lemma 2.3 [13]. (Discrete Jensen's Inequality). *Let f be a positive and convex function on $\mathbf{R}^+$. Then, for y_m and $\phi_m > 0$, $m\in \Omega$,*

$$\sum_{m\in\Omega} f(y_m)\phi_m \geq \sum_{m\in\Omega} \phi_m f\left(\frac{1}{\sum\limits_{m\in\Omega} \phi_m} \sum_{m\in\Omega} y_m \phi_m \right).$$

Theorem 2.1: *Assume that (h1)-(h4) hold. Furthermore, assume that there exists a positive sequence $\{\rho_n\}_{n=n_0}^{\infty}$ such that*

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[k\rho_l (1 - p_{l-\sigma}) q_l - \frac{(a_{l-\sigma})\,(\Delta\rho_l)^2}{4\rho_l} \right] = \infty. \tag{2.1}$$

Then, every solution of IBVPB1 is oscillatory in $\Omega \times \mathbf{N}_{n_0}$.

Proof: Suppose to the contrary that $\{y_{m,n}\}$ is a nonoscillatory solution of IBVPB1. Without loss of generality, we may assume that there exists $n_1 \in \mathbf{N}_{n_0}$ such that $y_{m,n-M} > 0$ for all $n \in \mathbf{N}_{n_1}$. Multiplying (1.1) by the eigenfunction ϕ_m and summing over m, we get

$$\begin{aligned} &\Delta_2(a_n \Delta_2(\sum_{m\in\Omega} \phi_m y_{m,n} + p_n \sum_{m\in\Omega} \phi_m y_{m,n-\tau})) + \sum_{m\in\Omega} q_{m,n} f(y_{m,n-\sigma})\phi_m \\ = \ & r_n \sum_{m\in\Omega} \phi_m L y_{m,n} + \sum_{j\in J} R_{j,n} \sum_{m\in\Omega} \phi_m L y_{m,n-\gamma_j}, \text{ for } n \in \mathbf{N}_{n_1}. \end{aligned} \tag{2.2}$$

From Lemma 2.2 and (B1) we find that

$$\begin{aligned} &\sum_{m\in\Omega} \phi_m L y_{m,n} - \sum_{m\in\Omega} y_{m,n} L \phi_m \\ = \ & \sum_{i=1}^{l} \left\{ \sum_{m_1=1}^{N_1} ... \sum_{m_{i-1}=1}^{N_{i-1}} \sum_{m_{i+1}=1}^{N_{i+1}} ... \sum_{m_l=1}^{N_l} [\phi_m \Delta_{m_i} y_{m,n} - y_{m,n} \Delta_{m_i} \phi_m]_{m_i=0}^{N_i} \right\} \\ = \ & \sum_{i=1}^{l} \{ \sum_{m_1=1}^{N_1} ... \sum_{m_{i-1}=1}^{N_{i-1}} \sum_{m_{i+1}=1}^{N_{i+1}} ... \sum_{m_l=1}^{N_l} (\phi_{m_1,m_2,...,m_{i-1},N_i,m_{i+1}...,m_l} \\ & \times[-y_{m_1,m_2,...,m_{i-1},N_i,m_{i+1}...,m_l,n}] - y_{m_1,m_2,...,m_{i-1},N_i,m_{i+1}...,m_l,n} \\ & [-\phi_{m_1,m_2,...,m_{i-1},N_i,m_{i+1}...,m_l}]) \Big\} = 0. \end{aligned} \tag{2.3}$$

Subsequently, in view of Lemma 2.1 relation (2.3) further reduces to

$$\sum_{m\in\Omega}\phi_m L y_{m,n}=\sum_{m\in\Omega}y_{m,n}L\phi_m=-\alpha_0\sum_{m\in\Omega}y_{m,n}\phi_m. \tag{2.4}$$

Similarly, for each $j\in\mathbf{J}$, we find that

$$\sum_{m\in\Omega}\phi_m L y_{m,n-\gamma_j}=\sum_{m\in\Omega}y_{m,n-\gamma_j}L\phi_m=-\alpha_0\sum_{m\in\Omega}y_{m,n-\gamma_j}\phi_m. \tag{2.5}$$

Further, an application of Lemma 2.3 and (h3) provide

$$\sum_{m\in\Omega}q_{m,n}f(y_{m,n-\sigma})\phi_m\geq q_n\sum_{m\in\Omega}\phi_m f\left(\frac{1}{\sum\limits_{m\in\Omega}\phi_m}\sum_{m\in\Omega}\phi_m y_{m,n-\sigma}\right),\ \text{for } n\in\mathbf{N}_{n_1}. \tag{2.6}$$

Set

$$u_n=\frac{1}{\sum\limits_{m\in\Omega}\phi_m}\sum_{m\in\Omega}\phi_m y_{m,n}. \tag{2.7}$$

Thus (h4), (2.2), (2.4)-(2.7) imply that

$$\Delta(a_n\Delta(u_n+p_nu_{n-\tau}))+kq_nu_{n-\sigma}\leq 0, \tag{2.8}$$

where Δ is the ordinary difference operator. Set

$$x_n=u_n+p_nu_{n-\tau}. \tag{2.9}$$

By, assumption (h1) we have $x_n>0$ for n$\geq n_1$ and from (2.7) it follows that

$$\Delta(a_n\Delta x_n)\leq -kq_nu_{n-\sigma}\leq 0,\qquad n\geq n_1 \tag{2.10}$$

and so $\{a_n\Delta x_n\}$ is an eventually nonincreasing sequence. We first show that $\Delta x_n\geq 0$ for $n\geq n_1$. In fact, if there exists an integer $n_2\geq n_1$ such that $a_{n_2}\Delta x_{n_2}=c<0$, then $a_n\Delta x_n\leq c$ for n$\geq n_2$ that is

$$\Delta x_n\leq\frac{c}{a_n}$$

and, hence by (h1) we have

$$x_n\leq x_{n_2}+c\sum_{i=n_2}^{n-1}\frac{1}{a_i}\to-\infty\ \ \text{as n}\to\infty$$

which contradicts the fact that $x_n>0$ for $n\geq n_1$. Therefore, we have

$$x_n>0,\ \Delta x_n\geq 0,\ \Delta(a_n(\Delta x_n))\leq 0,\ \ n\in\mathbf{N}_{n_1}. \tag{2.11}$$

Then, from (2.9) and (2.11) we find that $u_n\geq(1-p_n)x_n$ and this implies that for $n\geq n_2=n_1+\sigma$

$$u_{n-\sigma}\geq(1-p_{n-\sigma})x_{n-\sigma}.$$

From (2.10) and the last inequality, we have

$$\Delta(a_n \Delta x_n) + Q_n x_{n-\sigma} \leq 0, \quad n \in \mathbf{N}_{n_2}. \tag{2.12}$$

where $Q_n = kq_n(1 - p_{n-\sigma})$. Define the sequence $\{w_n\}$ by

$$w_n = \rho_n \frac{a_n \Delta x_n}{x_{n-\sigma}}. \tag{2.13}$$

Then $w_n > 0$, and

$$\Delta w_n = a_{n+1} \Delta x_{n+1} \Delta \left[\frac{\rho_n}{x_{n-\sigma}}\right] + \frac{\rho_n \Delta(a_n \Delta x_n)}{x_{n-\sigma}}. \tag{2.14}$$

Then (2.12) and (2.14), imply that

$$\Delta w_n \leq -\rho_n Q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n p_{n+1} \Delta x_{n+1} \Delta x_{n-\sigma}}{x_{n+1-\sigma} x_{n-\sigma}}. \tag{2.15}$$

But from (2.11), we have

$$a_{n-\sigma} \Delta x_{n-\sigma} \geq a_{n+1} \Delta x_{n+1}, \quad \textit{and} \quad x_{n+1-\sigma} \geq x_{n-\sigma}, \tag{2.16}$$

and thus from (2.15) and (2.16), we obtain

$$\Delta w_n \leq -\rho_n Q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\left(\rho_{n+1}\right)^2} \frac{1}{(a_{n-\sigma})} w_{n+1}^2. \tag{2.17}$$

So that

$$\begin{aligned} \Delta w_n \quad \leq \quad & -\rho_n Q_n + \frac{(a_{n-\sigma})\left(\Delta \rho_n\right)^2}{4\rho_n} - \\ & \left[\frac{\sqrt{\rho_n}}{\rho_{n+1}\sqrt{(a_{n-\sigma})}} w_{n+1} - \frac{\sqrt{(a_{n-\sigma})}\Delta \rho_n}{2\sqrt{\rho_n}}\right]^2 < -\left[\rho_n q_n - \frac{(a_{n-\sigma})\left(\Delta \rho_n\right)^2}{4\rho_n}\right]. \end{aligned}$$

Then, we have

$$\Delta w_n < -\left[\rho_n Q_n - \frac{(a_{n-\sigma})\left(\Delta \rho_n\right)^2}{4\rho_n}\right].$$

Summing the last inequality from n_2 to n, we obtain

$$-w_{n_2} < w_{n+1} - w_{n_2} < -\sum_{l=n_2}^{n} \left[\rho_l Q_l - \frac{(a_{l-\sigma})\left(\Delta \rho_l\right)^2}{4\rho_l}\right],$$

which yields

$$\sum_{l=n_2}^{n} \left[\rho_l Q_l - \frac{(a_{l-\sigma})\left(\Delta \rho_l\right)^2}{4\rho_l}\right] < c_1.$$

for all large n. This is contrary to (2.2). The proof is complete.

In Eq.(1.1) if $p_n = 0$, then Eq.(1.1) reduces to the hyperbolic delay difference equation

$$\Delta_2(a_n\Delta_2 y_{m,n}) + q_{m,n}f(y_{m,n-\sigma}) = r_n L y_{m,n} + \sum_{j\in J} R_{j,n} L y_{m,n-\gamma_j}. \tag{2.18}$$

Then from Theorem 2.1 we have the following corollary, which can be considered as the discrete analogy of Theorem 2.1 in [20] for hyperbolic delay differential equations.

Corollary 2.1: *Assume that (h1)-(h4) hold. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=n_0}^{\infty}$ *such that,*

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[k\rho_l q_l - \frac{(a_{l-\sigma})\,(\Delta\rho_l)^2}{4\rho_l} \right] = \infty.$$

Then, every solution of (2.18), (B1), (1.5) is oscillatory in $\Omega \times \mathbf{N}_{n_0}$.

From Theorem 2.1, we can obtain different conditions for oscillation of all solutions of IBVPB1 different choices of $\{\rho_n\}$. Let $\rho_n = n^\lambda$, n$\geq n_0$ and $\lambda > 1$ is a constant. By theorem 2.1 we have the following result.

Corollary 2.2: *Assume that all the assumption of Theorem 2.1 hold, except that the condition (2.1) is replaced by*

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \left[s^\lambda Q_s - \frac{(a_{s-\sigma})\,((s+1)^\lambda - s^\lambda)^2}{4s^\lambda} \right] = \infty.$$

Then, every solution of IBVPB1 is oscillatory in $\Omega \times \mathbf{N}_{n_0}$.

The following theorem can be considered as the discrete analogy of the Kamenev-type condition (1.7).

Theorem 2.2. *Assume that (h1)-(h4) hold. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=n_0}^{\infty}$ *such that for every* $\lambda \geq 1$,

$$\lim_{m\to\infty} \sup \frac{1}{m^\lambda} \sum_{n=n_1}^{m-1} (m-n)^\lambda \left[\rho_n Q_n - \frac{a_{n-\sigma}\left(\rho_{n+1}\right)^2}{4\rho_n} \left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^\lambda} \right)^2 \right] = \infty. \tag{2.19}$$

Then, every solution of IBVPB1 is oscillatory in $\Omega \times \mathbf{N}_{n_0}$.

Proof: We proceed as in the proof of Theorem 2.1, we assume that the IBVPB1 has a nonoscillatory solution $\{y_{m,n}\}$ in $\Omega \times \mathbf{N}_{n_0}$. Without loss of generality we may assume that it has a positive solution $\{y_{m,n-M}\}$ for $n \geq n_1$. Then we have $x_n > 0$, $\Delta x_n > 0$, $\Delta(a_n\,(\Delta x_n)) \leq 0$ for $n \geq n_1$. Define $\{w_n\}$ by (2.13) as before, then we have $w_n > 0$ for $n \geq n_2$ and (2.17) holds. For the sake of convenience, let us set $\bar{\rho}_n = \frac{\rho_n}{a_{n-\sigma}}$. Then, from (2.17) we obtain

$$\rho_n Q_n \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\bar{\rho}_n}{\rho_{n+1}^2} w_{n+1}^2.$$

Therefore,

$$\sum_{n=n_2}^{m-1}(m-n)^\lambda \rho_n Q_n \leq -\sum_{n=n_2}^{m-1}(m-n)^\lambda \Delta w_n + \sum_{n=n_2}^{m-1}(m-n)^\lambda \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_2}^{m-1}(m-n)^\lambda \frac{\bar{\rho}_n}{\rho_{n+1}^2} w_{n+1}^2. \tag{2.20}$$

Now, after summing by parts, we have

$$\sum_{n=n_2}^{m-1}(m-n)^\lambda \Delta w_n = -(m-n_2)^\lambda w_{n_2} - \sum_{n=n_2}^{m-1} w_{n+1}\Delta_2(m-n)^\lambda,$$

where, $\Delta_2(m-n)^\lambda = (m-n-1)^\lambda - (m-n)^\lambda$, then we have

$$\sum_{n=n_2}^{m-1}(m-n)^\lambda \Delta w_n = -(m-n_2)^\lambda w_{n_2} + \sum_{n=n_2}^{m-1} w_{n+1}((m-n)^\lambda - (m-n-1)^\lambda).$$

Using the inequality,

$$x^\beta - y^\beta \geq \beta y^{\beta-1}(x-y) \text{ for all x} \geq y > 0 \text{ and } \beta \geq 1,$$

we have

$$\sum_{n=n_2}^{m-1}(m-n)^\lambda \Delta w_n \geq -(m-n_2)^\lambda w_{n_2} + \sum_{n=n_2}^{m-1} \lambda w_{n+1}(m-n-1)^{\lambda-1}.$$

Substitute in (2.20), we have

$$\sum_{n=n_2}^{m-1}(m-n)^\lambda \rho_n Q_n \leq (m-n_2)^\lambda w_{n_2} - \sum_{n=n_2}^{m-1} \lambda w_{n+1}(m-n-1)^{\lambda-1} + \sum_{n=n_2}^{m-1}(m-n)^\lambda \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_2}^{m-1}(m-n)^\lambda \frac{\bar{\rho}_n}{\rho_{n+1}^2} w_{n+1}^2.$$

Then,

$$\begin{aligned}
&\frac{1}{m^\lambda}\sum_{n=n_2}^{m-1}(m-n)^\lambda \rho_n Q_n \\
\leq\ & (\frac{m-n_2}{m})^\lambda w_{n_2} - \frac{1}{m^\lambda}\sum_{n=n_2}^{m-1}(m-n)^\lambda \left[\frac{\bar{\rho}_n}{\rho_{n+1}^2} w_{n+1}^2 - \left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^\lambda}\right) w_{n+1}\right] \\
=\ & (\frac{m-n_2}{m})^\lambda w_{n_2} \\
& -\frac{1}{m^\lambda}\sum_{n=n_2}^{m-1}(m-n)^\lambda \left[\frac{\sqrt{\bar{\rho}_n}}{\rho_{n+1}} w_{n+1} - \frac{\rho_{n+1}}{2\sqrt{\bar{\rho}_n}}\left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^\lambda}\right)\right]^2 \\
& +\frac{1}{m^\lambda}\sum_{n=n_2}^{m-1}(m-n)^\lambda \frac{(\rho_{n+1})^2}{4\bar{\rho}_n}\left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^\lambda}\right)^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{m^{\lambda}}\sum_{n=n_2}^{m-1}(m-n)^{\lambda}\rho_n Q_n \\
&< \ (\frac{m-n_2}{m})^{\lambda} w_{n_2} + \frac{1}{m^{\lambda}}\sum_{n=n_2}^{m-1}(m-n)^{\lambda}\frac{\left(\rho_{n+1}\right)^2}{4\bar{\rho}_n}\left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)^2,
\end{aligned}$$

or

$$\frac{1}{m^{\lambda}}\sum_{n=n_2}^{m-1}(m-n)^{\lambda}\left[\rho_n Q_n - \frac{\left(\rho_{n+1}\right)^2}{4\bar{\rho}_n}\left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)^2\right] < (\frac{m-n_2}{m})^{\lambda} w_{n_2},$$

which yields

$$\lim_{m\to\infty}\frac{1}{m^{\lambda}}\sum_{n=n_2}^{m-1}(m-n)^{\lambda}\left[\rho_n Q_n - \frac{\left(\rho_{n+1}\right)^2}{4\bar{\rho}_n}\left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)^2\right] < \infty,$$

which is contrary to (2.19). The proof is complete.

Corollary 2.3. *Assume that all the assumptions of Theorem 2.2 hold, except that the condition (2.19) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\rho_n Q_n = \infty,$$

and

$$\lim_{m\to\infty}\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)\frac{a_{n-\sigma}\left(\rho_{n+1}\right)^2}{\rho_n}\left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)^2 < \infty.$$

Then, every solution of IBVPB1 is oscillatory in $\Omega \times \mathbf{N}_{n_0}$.

The following theorem can be considered as the discrete analogy of Philos-type condition (1.8).

Theorem 2.3. *Assume that (h1)-(h4) hold.* Let $\{\rho_n\}_{n=n_0}^{\infty}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *such that (i)* $H_{m,m} = 0$ *for* $m \geq 0$, *(ii)* $H_{m,n} > 0$ *f*or $m > n \geq 0$, (iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ for $m \geq n \geq 0$. *If*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n Q_n - \frac{\left(a_{n-\sigma}\right)\rho_{n+1}^2}{\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] = \infty. \tag{2.21}$$

where

$$h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}},\ m > n \geq 0, \tag{2.22}$$

Then, every solution of IBVPB1 is oscillatory in $\Omega \times \mathbf{N}_{n_0}$.

Proof: We proceed as in the proof of Theorem 2.1, we may assume that IBVPB1 has a nonoscillatory solution, without loss of generality, we assume that it has a positive solution in $\Omega\times\mathbf{N}_{n_0}$. Follows the proof of Theorem 2.1 we have $x_n > 0$, $\Delta x_n > 0$, $\Delta(a_n(\Delta x_n)) \leq 0$ for $n \geq n_1$. Define $\{w_n\}$ by (2.13) as before, then we have $w_n > 0$ for n$\geq n_2$ and (2.17) holds. Thus

$$\rho_n Q_n \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\bar{\rho}_n}{\left(\rho_{n+1}\right)^2} w_{n+1}^2. \tag{2.23}$$

Therefore, we have

$$\sum_{n=n_2}^{m-1} H_{m,n}\rho_n Q_n \leq -\sum_{n=n_2}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_2}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_2}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{\left(\rho_{n+1}\right)^2} w_{n+1}^2. \tag{2.24}$$

which yields, after summing by parts

$$\begin{aligned}
&\sum_{n=n_2}^{m-1} H_{m,n}\rho_n Q_n \\
\leq\ & H_{m,n_2} w_{n_2} + \sum_{n=n_2}^{m-1} w_{n+1}\Delta_2 H_{m,n} + \sum_{n=n_2}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_2}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{\left(\rho_{n+1}\right)^2} w_{n+1}^2 \\
=\ & H_{m,n_2} w_{n_2} - \sum_{n=n_2}^{m-1} h_{m,n}\sqrt{H_{m,n}} w_{n+1} + \sum_{n=n_2}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\
& - \sum_{n=n_2}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{\left(\rho_{n+1}\right)^2} w_{n+1}^2 = H_{m,n_2} w_{n_2} \\
& - \sum_{n=n_2}^{m-1}\left[\frac{\sqrt{H_{m,n}\ \bar{\rho}_n}}{\rho_{n+1}} w_{n+1} + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}\ \bar{\rho}_n}}\left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}} H_{m,n}\right)\right]^2 \\
& + \frac{1}{4}\sum_{n=n_2}^{m-1}\frac{\left(\rho_{n+1}\right)^2}{\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2.
\end{aligned}$$

Then,

$$\sum_{n=n_2}^{m-1}\left[H_{m,n}\rho_n Q_n - \frac{\left(\rho_{n+1}\right)^2}{4\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,n_2} w_{n_2} \leq H_{m,0} w_{n_2},$$

which implies that

$$\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n Q_n - \frac{\left(\rho_{n+1}\right)^2}{4\ \bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,0}\left(w_{n_2} + \sum_{n=0}^{n_2-1} n_2\rho_n q_n\right).$$

Hence

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\rho_nQ_n-\frac{\left(\rho_{n+1}\right)^2}{4\,\bar{\rho}_n}\left(h_{m,n}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right]$$
$$<\quad\left(w_{n_2}+\sum_{n=0}^{n_2-1}\rho_nQ_n\right)<\infty,$$

which is contrary to (2.21). The proof is complete.

As an immediate consequence of Theorem 2.3, we get the following:

Corollary 2.4: *Assume that all the assumptions of Theorem 2.3 hold, except that the condition (2.20) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1}H_{m,n}\rho_nQ_n=\infty,$$

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1}\frac{\left(a_{n-\sigma}\right)\rho_{n+1}^2}{\rho_n}\left(h_{m,n}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2<\infty.$$

Then, every solution of IBVPB1 is oscillatory in $\Omega\times\mathbf{N}_{n_0}$.

By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$\left.\begin{array}{l} H_{m,n}=(m-n)^{\lambda},\quad \lambda\geq 1, m\geq n\geq 0,\\ H_{m,n}=\left(\log\frac{m+1}{n+1}\right)^{\lambda}, \lambda\geq 1, m\geq n\geq 0,\\ H_{m,n}=(m-n)^{(\lambda)}\quad \lambda>2,\ m\geq n\geq 0. \end{array}\right\}\tag{2.25}$$

where $(m-n)^{(\lambda)}=(m-n)(m-n+1)...(m-n+\lambda-1)$, and

$$\Delta_2(m-n)^{(\lambda)}=(m-n-1)^{(\lambda)}-(m-n)^{(\lambda)}=-\lambda(m-n)^{(\lambda-1)}.$$

Then $H_{m,m}=0$ for m$\geq$ 0 and $H_{m,n}>0$ and $\Delta_2H_{m,n}\leq 0$ for m>n>$\geq$ 0. Hence we have the following results.

Corollary 2.5. *Assume that all the assumptions of Theorem 2.3 hold, except that the condition (2.21) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{\lambda}}\sum_{n=0}^{m-1}\left[(m-n)^{\lambda}\rho_nQ_n-\frac{\rho_{n+1}^2}{4\,\bar{\rho}_n}\left(\lambda(m-n)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{(m-n)^{\lambda}}\right)^2\right]=\infty.$$

Then, every solution of IBVPB1 is oscillatory in $\Omega\times\mathbf{N}_{n_0}$.

Corollary 2.6. *Assume that all the assumptions of Theorem 2.3 hold, except that the condition (2.21) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{(\log(m+1))^{\lambda}}\sum_{n=0}^{m-1}\left[\left(\log\frac{m+1}{n+1}\right)^{\lambda}\rho_nQ_n\right.$$

$$-\frac{\rho_{n+1}^2}{4\bar{\rho}_n}\left(\frac{\lambda}{n+1}\left(\log\frac{m+1}{n+1}\right)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{\left(\log\frac{m+1}{n+1}\right)^{\lambda}}\right)^2\Bigg]=\infty.$$

Then, every solution of IBVPB1 is oscillatory in $\Omega\times\mathbf{N}_{n_0}$.

Corollary 2.7. *Assume that all the assumptions of Theorem 2.3 hold, except that the condition (2.21) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{(\lambda)}}\sum_{n=0}^{m-1}(m-n)^{(\lambda)}\left[\rho_nQ_n-\frac{\rho_{n+1}^2}{4\bar{\rho}_n}\left(\frac{\lambda}{m-n+\lambda-1}-\frac{\Delta\rho_n}{\rho_{n+1}}\right)^2\right]=\infty.$$

Then, every solution of IBVPB1 is oscillatory in $\Omega\times\mathbf{N}_{n_0}$.

3 Oscillation of IBVPB2

In this section we will establish some new sufficient condition for oscillation of all solution of *IBVPB2*.

Theorem 3.1: *Assume that (h1)-(h4) hold. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=n_0}^{\infty}$ *such that (2.2) holds. Then, every solution of IBVPB2 is oscillatory in* $\Omega\times\mathbf{N}_{n_0}$.

Proof: Suppose to the contrary that $\{y_{m,n}\}$ is a nonoscillatory solution of IBVPB2. Without loss of generality, we may assume that there exists $n_1\in\mathbf{N}_{n_0}$ such that $y_{m,n-M}>0$ for all $n\in\mathbf{N}_{n_1}$. Summing Eq.(1.1) over m, we get

$$\begin{aligned}&\Delta_2(p_n\Delta_2(\sum_{m\in\Omega}y_{m,n}+p_n\sum_{m\in\Omega}y_{m,n-\tau}))+\sum_{m\in\Omega}q_{m,n}f(y_{m,n-\sigma})\\ =\;& r_n\sum_{m\in\Omega}Ly_{m,n}+\sum_{j\in J}R_{j,n}\sum_{m\in\Omega}Ly_{m,n-\gamma_j},\text{ for n}\in\mathbf{N}_{n_1}.\end{aligned}\tag{3.1}$$

Applying Lemma 2.2 and using (B2), we find that

$$\begin{aligned}\sum_{m\in\Omega}Ly_{m,n}&=\sum_{i=1}^{l}\left\{\sum_{m_1=1}^{N_1}...\sum_{m_{i-1}=1}^{N_{i-1}}\sum_{m_{i+1}=1}^{N_{i+1}}...\sum_{m_l=1}^{N_l}[\Delta_{m_i}y_m]_{m_i=0}^{N_i}\right\}\\ =&\sum_{i=1}^{l}\{\left\{\sum_{m_1=1}^{N_1}...\sum_{m_{i-1}=1}^{N_{i-1}}\sum_{m_{i+1}=1}^{N_{i+1}}...\sum_{m_l=1}^{N_l}[-g_{m,n}y_{m,n}|_{m_i=N_{i+1}}-g_{m,n}y_{m,n}|_{m_{ii}=0}]\right\}\\ =& A_{m,n}\le 0.\end{aligned}\tag{3.2}$$

Similarly, for each $j\in\mathbf{J}$ we find that

$$\sum_{m\in\Omega}Ly_{m,n-\gamma_j}=A_{m,n-\gamma_j}\le 0.\tag{3.3}$$

Further, an application of Lemma 2.3 provides

$$\sum_{m\in\Omega} q_{m,n} f(y_{m,n-\sigma}).1 \geq q_n \sum_{m\in\Omega} f\left(\frac{1}{|\Omega|}\sum_{m\in\Omega} y_{m,n-\sigma}\right)|\Omega|\,, \text{ for } n\in \mathbf{N}_{n_1}. \tag{3.4}$$

Set

$$u_n = \frac{1}{|\Omega|}\sum_{m\in\Omega} y_{m,n}, \tag{3.5}$$

where $|\Omega| = \sum_{m\in\Omega} 1 = N_1 N_2 ... N_l$. Thus (h4) and (3.1)-(3.5) imply that

$$\Delta(a_n \Delta(u_n + p_n u_{n-\tau})) + k q_n u_{n-\sigma} \leq 0, \tag{3.6}$$

where Δ is the ordinary difference operator. The remainder of the proof is similar to that of the proof of Theorem 2.1 and hence is omitted.

As in the proof of Theorems 2.2 and 2.3 by using (3.6) we can prove the following theorems.

Theorem 3.2. *Assume that (h1)-(h4) hold. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=0}^{\infty}$ *such that for every* $\lambda \geq 1$, *(2.18) holds. Then, every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbf{N}_{n_0}$.

Theorem 3.3. *Assume that (h1)-(h4) hold.* Let $\{\rho_n\}_{n=n_0}^{\infty}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *such that (i)* $H_{m,m} = 0$ *for* $m \geq 0$, *(ii)* $H_{m,n} > 0$ for $m > n \geq 0$, (iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ for $m \geq n \geq 0$. *If (2.20) holds, then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbf{N}_{n_0}$.

From Theorem 2.3 by using the definition of $\{H_{m,n}\}$ as in (2.25), corresponding corollaries for oscillation of IBVPB2 can be stated. The details are left to the reader.

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